General Relativistic Hydrodynamics

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Dynamics of Many Particle systems

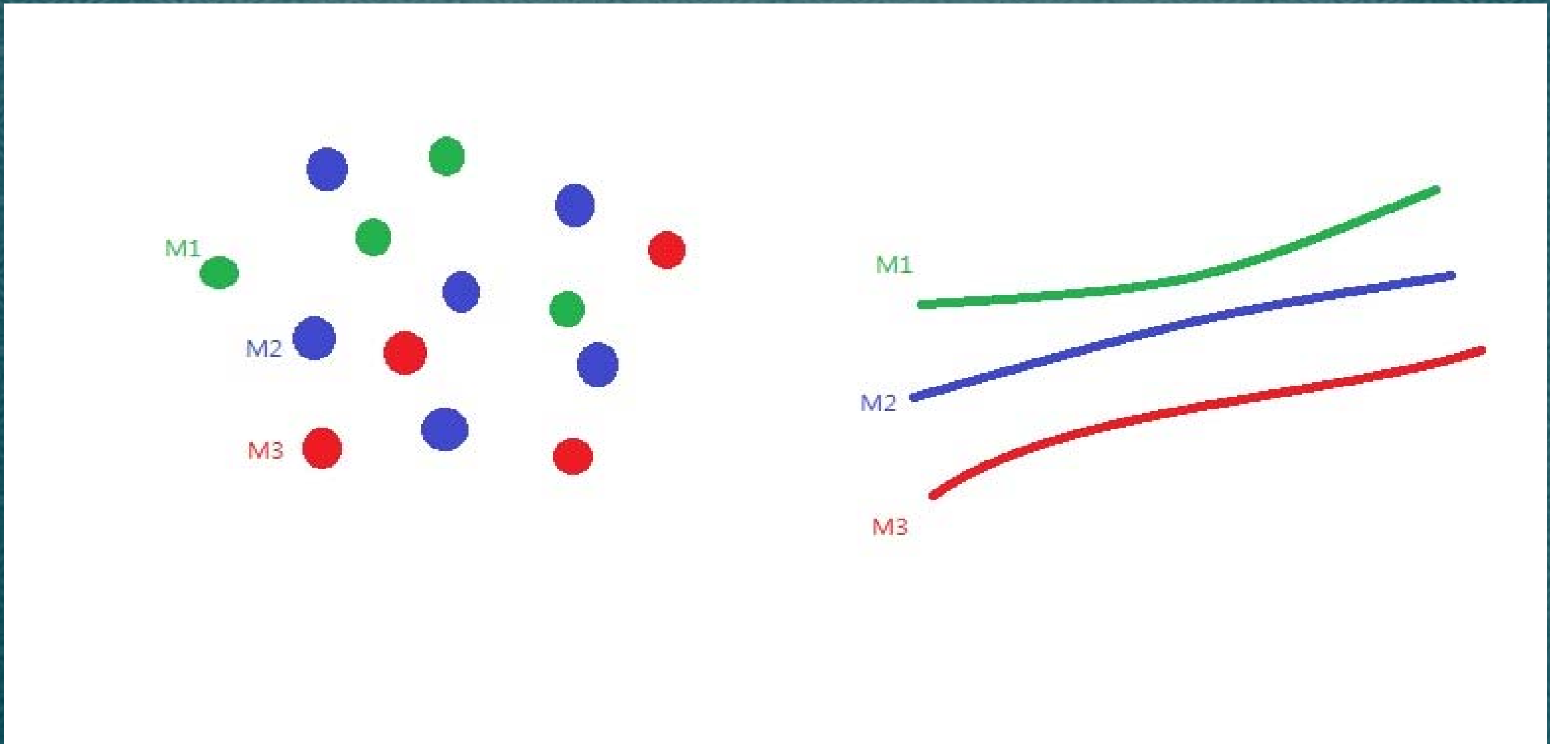
- N-Body

  M1, M2, M3, … = constants

- Hydrodynamics

  M1 = M1(x, t), etc

  mass density, pressure, …



Slide Courtesy

- J. Font (2009) APCTP Lecture
- Whisky, http://whiskycode.org

GRHydro

- GRHydro eqs $T^{\mu\nu}{}_{;\nu} = 0 \ \ \& \ \ (\rho u^{\mu})_{;\mu}$

  with EoS p=p(ρ,ε) $$T^{\mu\nu} = \rho\left(1+\epsilon+\frac{p}{\rho}\right)u^{\mu}u^{\nu} + pg^{\mu\nu}$$

- High Resolution Shock Capturing (HRSC)
  - Cf. MUSCL for 2nd order HRSC ???
  - Flux conservative method for GR: Valencia formulation
    - Need to redefine the hydro variables: Primitive vs Conservative
  - Method of lines
  - Reconstruction
  - Flux calculation

Hyperbolic systems of conservation law

- A is const matrix
- w is characteristic
- Lambda is diagonal eigenvalue matrix
- Advection-type solution

$$\frac{\partial \vec{u}}{\partial t} + A\frac{\partial \vec{u}}{\partial x} = 0$$

$$\vec{w} = R^{-1}\vec{u}$$

$$\frac{\partial \vec{w}}{\partial t} + \Lambda\frac{\partial \vec{w}}{\partial x} = 0$$

$$w_i(x,t) = w_i(x - \lambda_i t, 0)$$

$$\vec{u}(x,t) = R\vec{w}(x,t)$$

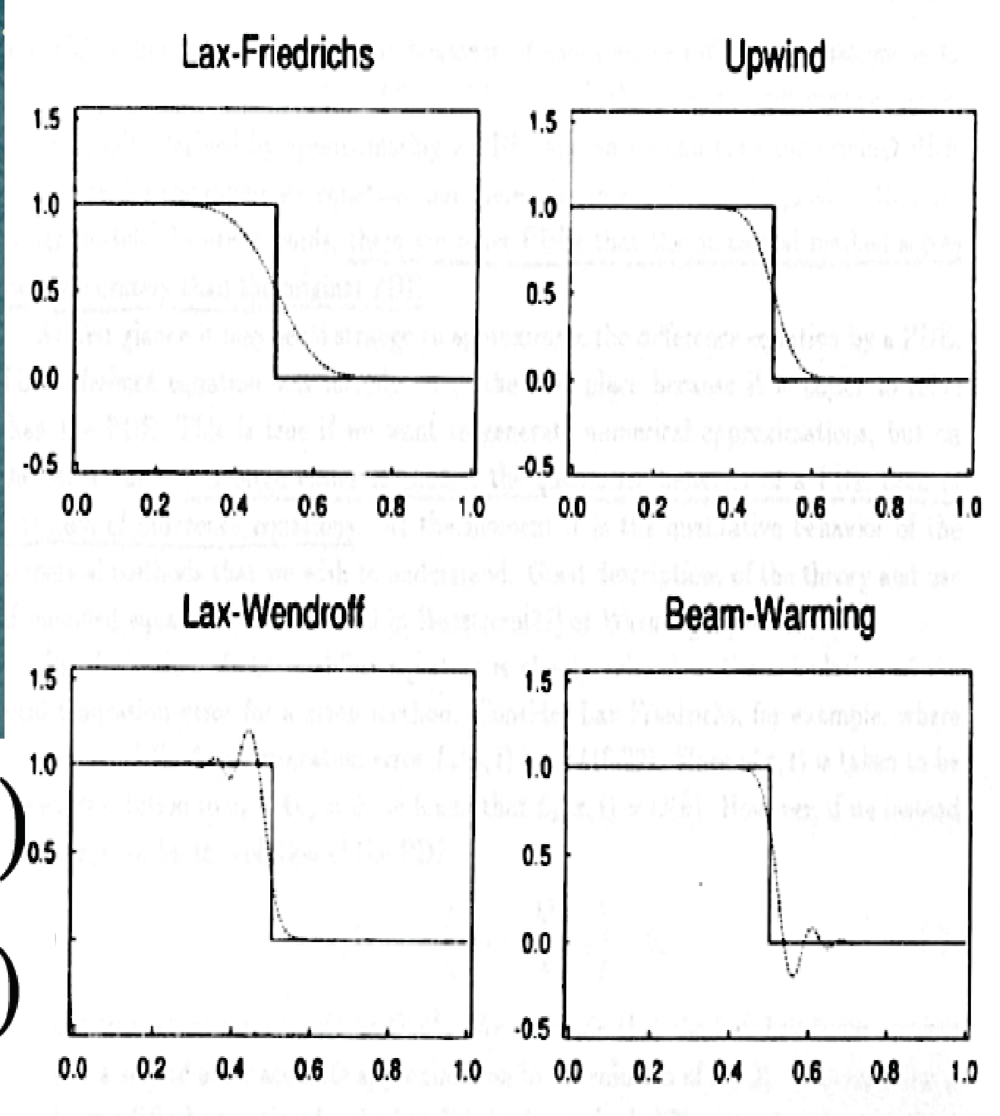
Simple finite difference method

1st order:
diffused

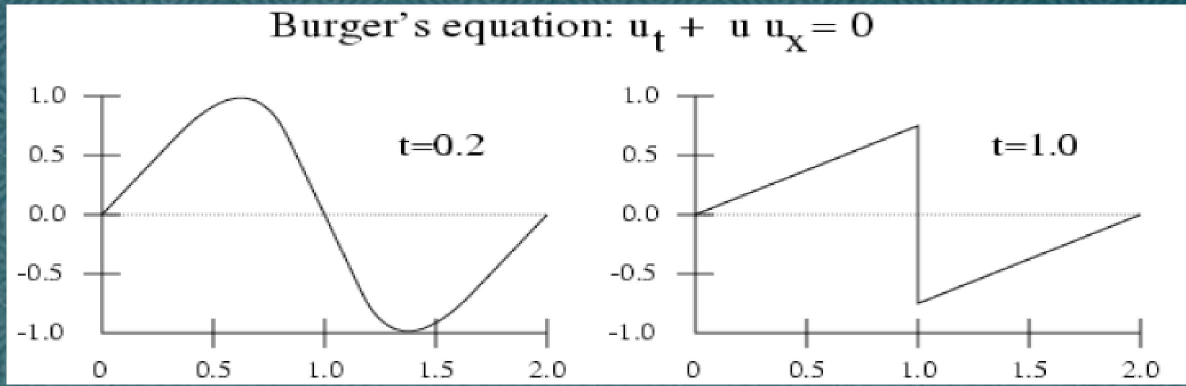
$$w_j^{n+1} = w_j^n - \frac{\Delta t}{\Delta x}\lambda\left(w_{j+1}^n - w_j^n\right)$$

2nd order:
Spurious oscillations

$$w_j^{n+1} = w_j^n - \frac{\Delta t}{2\Delta x}\lambda\left(w_{j+1}^n - w_{j-1}^n\right) + \frac{(\Delta t)^2}{2(\Delta x)^2}\lambda^2\left(w_{j+1}^n - 2w_j^n + w_{j-1}^n\right)$$

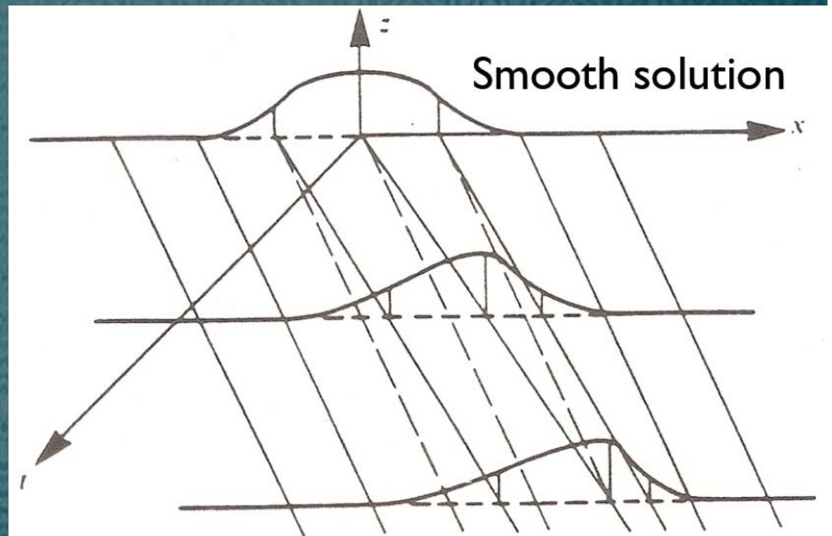


Non-linear system: Shock formation

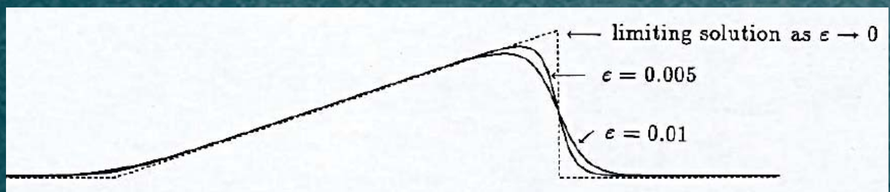


$$u_t + uu_x = 0$$

$$u_t + uu_x = \varepsilon u_{xx}$$



- Artificial viscosity may work but …



Finite volume method

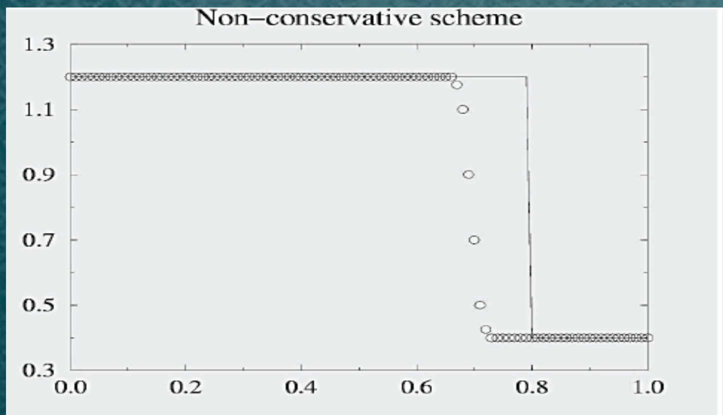
- Redefine variables by averaging over a fluid element
- Final equation is nothing but integral expression of the original equation.
- Finding a solution = calculation of flux through cell interfaces

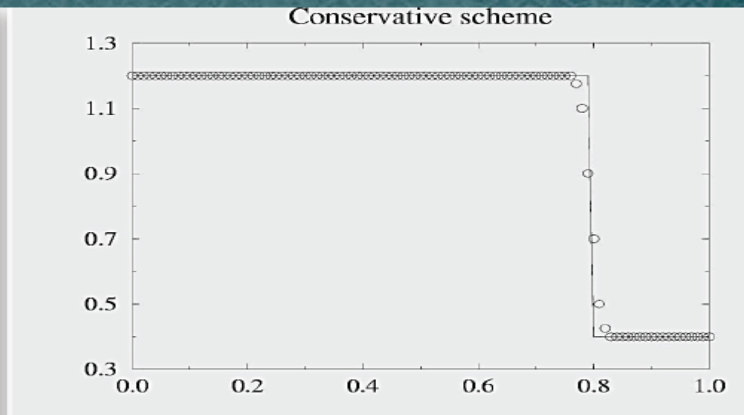
$$\vec{u}_j^n \approx \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \vec{u}(x,t^n)dx$$

$$\mathbf{u}_j^{n+1} = \mathbf{u}_j^n - \frac{\Delta t}{\Delta x}\left(\hat{\mathbf{f}}_{j+\frac{1}{2}}^n - \hat{\mathbf{f}}_{j-\frac{1}{2}}^n\right)$$

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}u_j^n(u_j^n - u_{j-1}^n)$$

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left(\frac{1}{2}(u_j^n)^2 - \frac{1}{2}(u_{j-1}^n)^2\right)$$





Flux conservative eqs of GR-Hydro

Primitive: ρ, p, $\mathbf{v}$, h, W → *Conservative:* D, S, τ

$$\partial_t \mathbf{q} + \partial_{x^i} \mathbf{f}^{(i)}(\mathbf{q}) = \mathbf{s}(\mathbf{q})$$

$$\begin{aligned} D &= \sqrt{\gamma} W \rho \\ S_j &= \sqrt{\gamma} \rho h W^2 v_j \\ \tau &= \sqrt{\gamma}\left(\rho h W^2 - p\right) - D, \end{aligned}$$

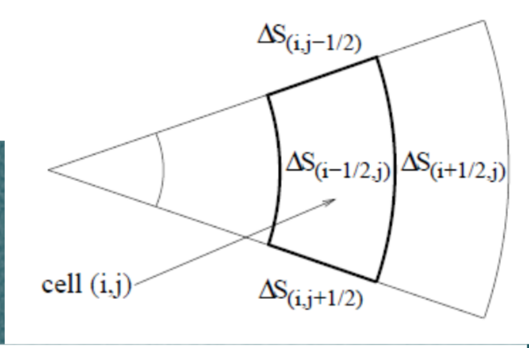
$$\mathbf{f}^{(i)} = \alpha \begin{pmatrix} D\left(v^i - \beta^i/\alpha\right) \\ S_j\left(v^i - \beta^i/\alpha\right) + \delta^i_j \sqrt{\gamma} p \\ \tau\left(v^i - \beta^i/\alpha\right) + \sqrt{\gamma} p v^i \end{pmatrix}$$

$$\mathbf{s} = \alpha\sqrt{\gamma} \begin{pmatrix} 0 \\ T^{\mu\nu} g_{\nu\sigma} \Gamma^{\sigma}_{\mu j} \\ T^{\mu 0} \partial_\mu \alpha - \alpha T^{\mu\nu} \Gamma^{0}_{\mu\nu} \end{pmatrix}$$

Method of Lines (MoL)

PDE → ODE by integrating over a cell

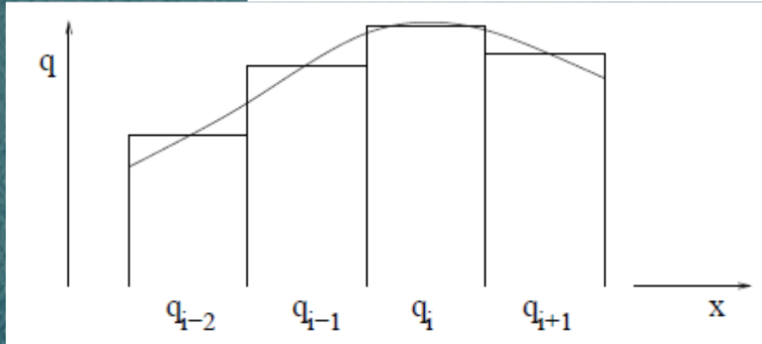
$$\int_{\Delta V} \frac{\partial \mathbf{q}}{\partial t}\, dV + \int_{\Delta V} \frac{\partial \mathbf{f}^i}{\partial x^i}\, dV = \frac{\partial}{\partial t}\int_{\Delta V} \mathbf{q}\, dV + \int_{\partial V} \mathbf{f}^i\, dS_i = 0$$



$$\mathbf{q}_{\vec{i}} = \frac{1}{\Delta V_{\vec{i}}} \int_{\Delta V_{\vec{i}}} \mathbf{q}\, dV$$

$$\frac{d\mathbf{q}_{\vec{i}}}{dt} + \frac{1}{\Delta V_{\vec{i}}} \int_{\partial \Delta V_{\vec{i}}} \mathbf{f}^i\, dS_i = 0$$

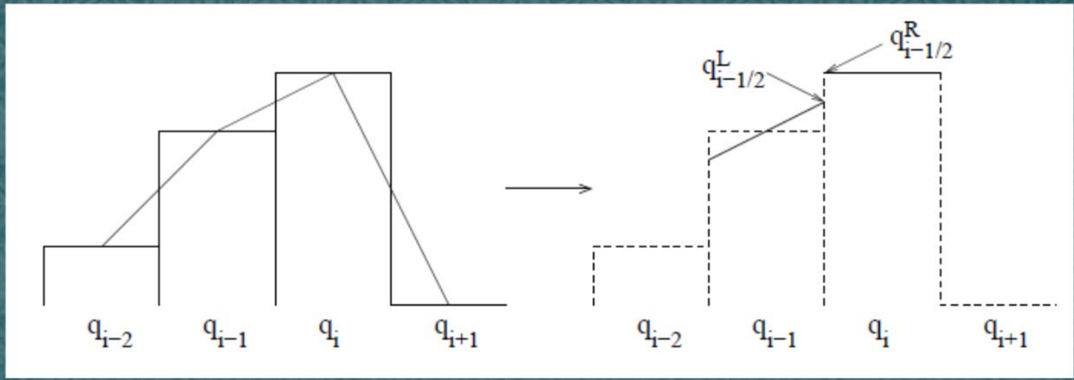
$$\frac{d\mathbf{q}_{\vec{i}}}{dt} + \frac{1}{\Delta V_{\vec{i}}} \sum_{\vec{l}} \mathbf{f}_{\vec{i}+\vec{l}} \Delta S_{\vec{i}+\vec{l}} = 0$$



Should Calculate the flux at the cell boundaries!

Reconstruction

: *to increase the order of accuracy*



- 1st order: Godunov
- 2nd order: Minmod (limiter):
  - MUSCL *Monotone Upstream-centered Schemes for Conservation Laws*
- 3rd order:

  PPM (Piecewise parabolic method)

  ENO (Essentially Non-Oscillatory)

Flux calculation

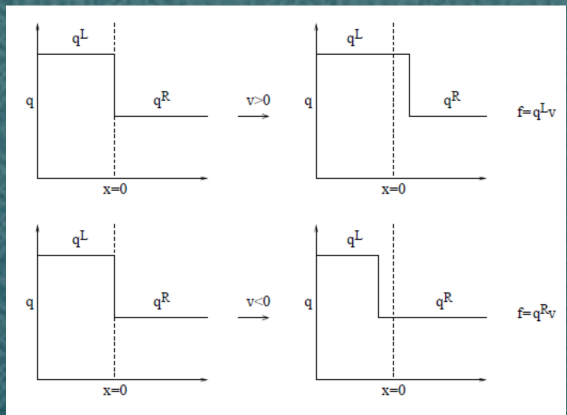
Conservative system with
Discontinuous initial data →

$$q_0(x) = \begin{cases} q^L & \text{if} \quad x < 0 \\ q^R & \text{if} \quad x > 0 \end{cases}$$

For linear eq.

$$\frac{\partial q}{\partial t} + v\frac{\partial q}{\partial x} = 0$$

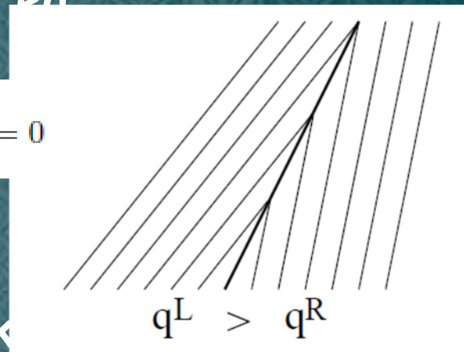
$$q(t,x) = q_0(x - vt)$$



Ex) nonlinear eq

$$\frac{\partial q}{\partial t} + \frac{\partial\left(\frac{1}{2}q^2\right)}{\partial x} = 0$$

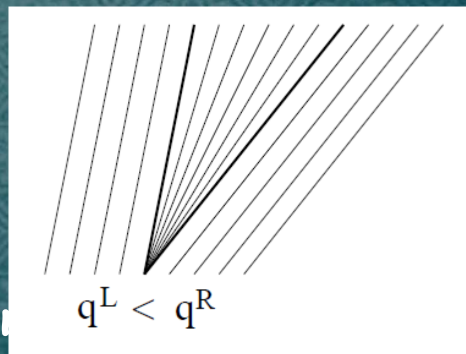
Shock



Nonlinear Solvers:

- Roe solver
- HLLE (Harten-Lax-van Leer-Einfeldt) solver
- Marquina solver

Rarefactio



Con2Prim & Atmosphere

- Con2Prim
  - Newton-Rahpson method
  - EoS is used
  - Most of NaNs comes from this part.
- Atmosphere
  - Vacuum has v_sound = c
  - Assign very small density
  - Usually having Polytropic EoS
  - Zero velocity

$$p - \bar{p}[\rho(\mathbf{q}, p), \varepsilon(\mathbf{q}, p)] = 0$$

$$\rho = \frac{D}{\tau + p + D}\sqrt{(\tau + p + D)^2 - S^2},$$

$$\varepsilon = D^{-1}\left[\sqrt{(\tau + p + D)^2 - S^2} - p\bar{W} - D\right],$$

where

$$\bar{W} = \frac{\tau + p + D}{\sqrt{(\tau + p + D)^2 - S^2}}$$

$$\frac{d}{dp}\{p - \bar{p}[\rho(\mathbf{q}, p), \varepsilon(\mathbf{q}, p)]\} = 1 - \frac{\partial \bar{p}(\rho, \varepsilon)}{\partial \rho}\frac{\partial \rho}{\partial p} - \frac{\partial \bar{p}(\rho, \varepsilon)}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial p},$$

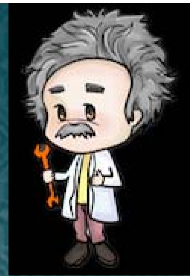
where

$$\frac{\partial \rho}{\partial p} = \frac{DS^2}{\sqrt{(\tau + p + D)^2 - S^2}(\tau + p + D)^2},$$

$$\frac{\partial \varepsilon}{\partial p} = \frac{pS^2}{\rho[(\tau + p + D)^2 - S^2](\tau + p + D)},$$

EinsteinToolkit

http://einsteintoolkit.org

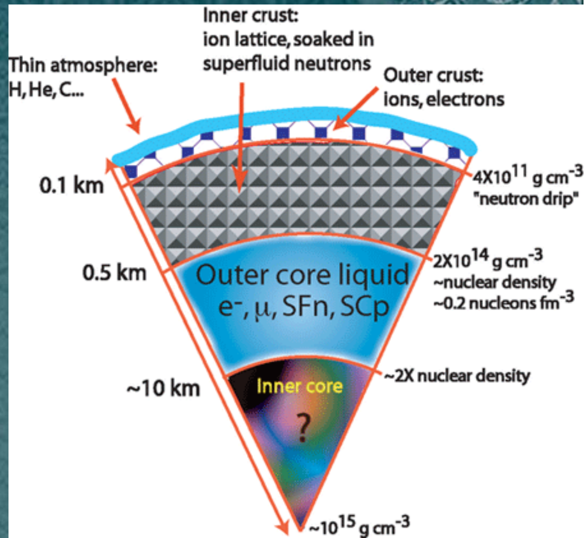


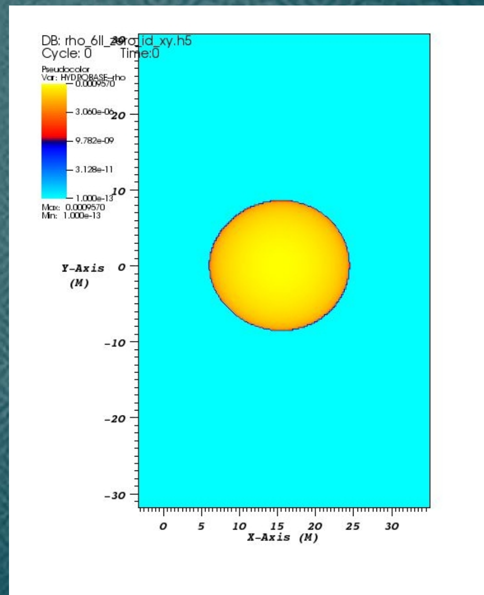
- Open software consortium for NR
- Written in
  - Cactus (open source problem solving environment)
    - Flesh + Thorn
    - Modularity
    - Automatic MPI parallelization cf. OpenMP, CacCUDA
- Other components
  - Carpet: AMR driver for Cactus
  - GRHydro: one branch of Whisky
  - McLachlan: BSSN code, CCZ4
  - Kranc: code generator
  - Simfactory: Job manager

- Additional Code of our group
  - Pseudo-Newtonian 2D Hydro (aka. Soju, MNRAS12)
  - 2D GRHydro code
    - WashU code + table EoS

Applications of GRHydro

- Neutron stars
  - Puzzling phenomena: SN core collapse, GRB, SGR, pulsar glitch, …
  - Strong gravitational wave sources: single, NS+NS, NS+BH (cf. mostly 3D phenomena)
  - Experimental lab for dense matter physics
- Supermassive stars
  - One possible origin of Supermassive black holes
- What is required for the study of relativistic stars?
  - GRHydro = General relativity + Hydrodynamics
  - Numerrical codes + Computational resources
- Current main goals
  - GW waveforms from single & binary NS sources
  - Possible EoS dependence
  - By implementing
    - MHD
    - Realistic EoS
    - Neutrino transport
  - Trivia: Collapse of Supermassive star, Early universe, …
- Cf. dynamical timescale ~ msec
  - Max simulated time < 0.1 sec
  - Highest resolution dx = 0.18km
  - Binary seperation = 60km





Einstein Equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -8\pi T_{\mu\nu}$$

- Second order PDE for 10 metric components = Energy-momentum (EM)
- 4 constraint eqs from Bianchi identity
- → 2^{nd} order PDEs for 6 unknowns + 4 gauge conditions
- Initial value problem
- Hydrodynamics is described by the matter eqs. of motion along with current conservation (e.g. baryon number)

$$T^{\mu\nu};\nu = 0 \quad \& \quad (\rho u^{\mu})_{;\mu}$$

- Highly nonlinear & coupled eqs
- ~ 8 byte x 300 variables x 600^3 grid points ~ 500 GB
- cf. BBH: (not all the time) (AMR) ~ 80 GB

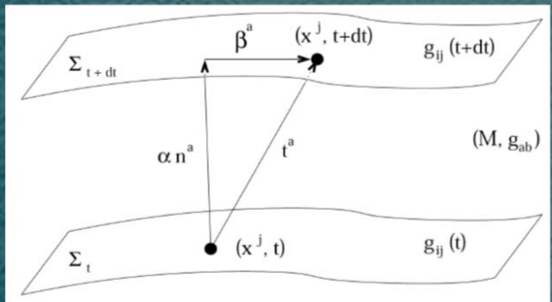
3+1 spacetime decomposition (Arnowitt, Deser, & Misner, ADM)

$$g_{\mu\nu} \to (\alpha, \beta^i, \gamma_{ij})$$

$$(g_{\mu\nu}, \partial g_{\mu\nu}) \to (\gamma_{ij}, K_{ij})$$

$$ds^2 = -\alpha\, dt^2 + \gamma_{ij}\left(\beta^i dt + dx^i\right)\left(\beta^j dt + dx^j\right)$$

$$K_{ij} = -\frac{1}{2} L_n \gamma_{ij}$$



$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i,$$

$$\partial_t K_{ij} = -\nabla_i \nabla_j \alpha + \alpha\Big[R_{ij} + K K_{ij} - 2K_{im}K^m_j - 8\pi\Big(S_{ij} - \frac{1}{2}\gamma_{ij} S\Big) - 4\pi \rho_{\mathrm{ADM}} \gamma_{ij}\Big] + \beta^m \nabla_m K_{ij} + K_{im} \nabla_j \beta^m + K_{mj} \nabla_i \beta^m$$

- γ_ij : 3-space metric
- α : lapse
- β_i : shift
- 10 metric comps → 3-space metric (6)
- \+ gauge condition (4)
- Constraint eqs are used to find initial data
- ADM method, however, is unstable

$$^{(3)}R + K^2 - K_{ij}K^{ij} - 16\pi \rho_{\mathrm{ADM}} = 0$$

$$\nabla_j K^{ij} - \gamma^{ij} \nabla_j K - 8\pi j^i = 0$$

BSSN formulation
(Baumgarte&Shapiro 98, Shibata&Nakamura 95)

- Introduce auxilary variables to remove unstable modes: make the eqs. strongly hyperbolic
- Most popular stable evolution method

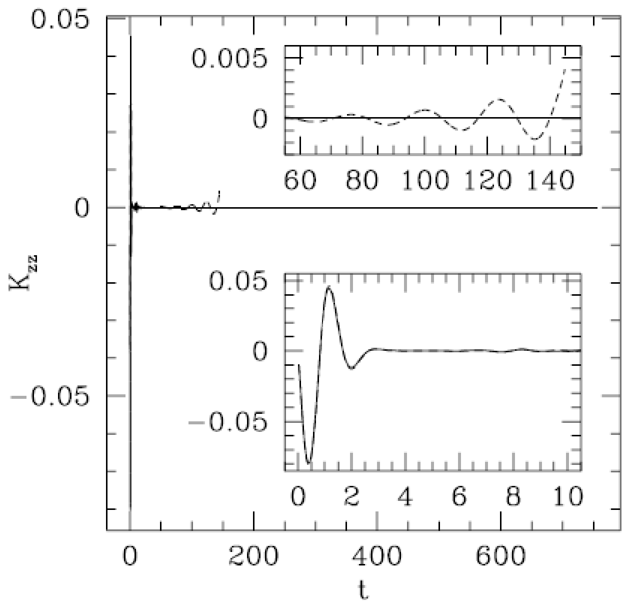
$$\varphi = (1/12)\log(\det\gamma_{ij}), \qquad \tilde{\gamma}_{ij} = e^{-4\varphi}\gamma_{ij}, \qquad K = \gamma^{ij}K_{ij},$$
$$\tilde{A}_{ij} = e^{-4\varphi}(K_{ij} - (1/3)\gamma_{ij}K), \qquad \tilde{\Gamma}^i = \tilde{\Gamma}^i_{jk}\tilde{\gamma}^{jk}.$$

- The evolution equations:

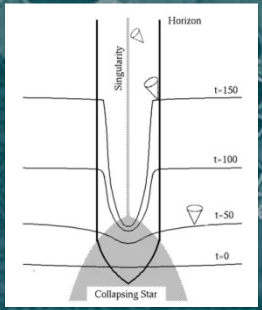
$$\begin{aligned}
\partial_t^B \varphi &= -(1/6)\alpha K + (1/6)\beta^i(\partial_i\varphi) + (\partial_i\beta^i),\\
\partial_t^B \tilde{\gamma}_{ij} &= -2\alpha\tilde{A}_{ij} + \tilde{\gamma}_{ik}(\partial_j\beta^k) + \tilde{\gamma}_{jk}(\partial_i\beta^k) - (2/3)\tilde{\gamma}_{ij}(\partial_k\beta^k) + \beta^k(\partial_k\tilde{\gamma}_{ij}),\\
\partial_t^B K &= -D^iD_i\alpha + \alpha\tilde{A}_{ij}\tilde{A}^{ij} + (1/3)\alpha K^2 + \beta^i(\partial_i K),\\
\partial_t^B \tilde{A}_{ij} &= -e^{-4\varphi}(D_iD_j\alpha)^{TF} + e^{-4\varphi}\alpha(R_{ij}^{BSSN})^{TF} + \alpha K\tilde{A}_{ij} - 2\alpha\tilde{A}_{ik}\tilde{A}^k{}_j\\
&\quad +(\partial_i\beta^k)\tilde{A}_{kj} + (\partial_j\beta^k)\tilde{A}_{ki} - (2/3)(\partial_k\beta^k)\tilde{A}_{ij} + \beta^k(\partial_k\tilde{A}_{ij}),\\
\partial_t^B \tilde{\Gamma}^i &= -2(\partial_j\alpha)\tilde{A}^{ij} + 2\alpha(\tilde{\Gamma}^i_{jk}\tilde{A}^{kj} - (2/3)\tilde{\gamma}^{ij}(\partial_j K) + 6\tilde{A}^{ij}(\partial_j\varphi))\\
&\quad -\partial_j(\beta^k(\partial_k\tilde{\gamma}^{ij}) - \tilde{\gamma}^{kj}(\partial_k\beta^i) - \tilde{\gamma}^{ki}(\partial_k\beta^j) + (2/3)\tilde{\gamma}^{ij}(\partial_k\beta^k)).
\end{aligned}$$

Constraint equations:

$$\begin{aligned}
\mathcal{H}^{BSSN} &= R^{BSSN} + K^2 - K_{ij}K^{ij},\\
\mathcal{M}_i^{BSSN} &= \mathcal{M}_i^{ADM},\\
\mathcal{G}^i &= \tilde{\Gamma}^i - \tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk},\\
\mathcal{A} &= \tilde{A}_{ij}\tilde{\gamma}^{ij},\\
\mathcal{S} &= \tilde{\gamma} - 1.
\end{aligned}$$



Gauge condition



- Freedom to choose the lapse and the shift
- Bad example: Gaussian normal coordinates; a = 1, b = 0
- Could avoid singularities, handles everything ok for long enough time
- Slicing condition
  - K-freezing condition
  - 1+log

$$\partial_t \alpha = -f(\alpha)\alpha^2(K - K_0)$$

- Shift condition
  - Gamma-freezing or Gamma-driver

$$\partial_t^2 \beta^i = F \partial_t \tilde{\Gamma}^i - \eta \partial_t \beta^i$$

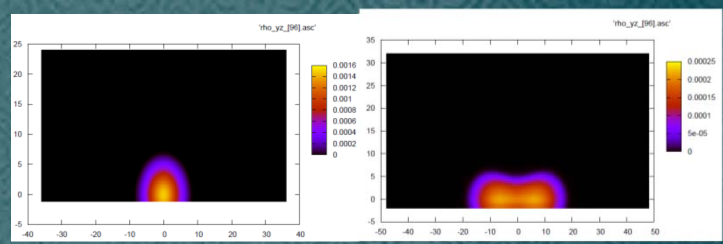
  - Moving puncture gauge

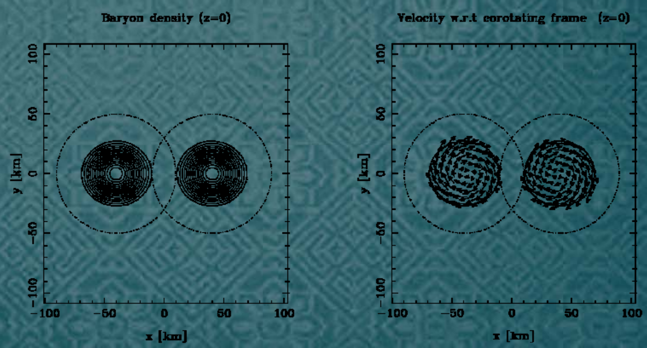
$$\partial_t \beta^i - \beta^j \partial_j \beta^i = \tfrac{3}{4} B^i,$$

$$\partial_t B^i - \beta^j \partial_j B^i = \partial_t \tilde{\Gamma}^i - \beta^j \partial_j \tilde{\Gamma}^i - \eta B^i$$

Initial data

- Found by solving the constraint eqs.
  - Constraint violation grows in general
  - Better to be accurate as much as possible
- Initial equilibrium is assumed in most cases
  - Single non-rotating star: TOV
  - Single rotating star: elliptic eqs.
    - Iterative self-consistent methods
    - RNS code by Stergigoulas & Morsink
      - Various EoS
  - Lorene code of Meudon group
    - Spectral method
    - Various EoS
    - Public code
    - Single (magnetized) rotating star
    - NS+NS (non-magnetized)
    - NS+NS (piecewise EoS): non-public
    - NS+BH(excision)
    - NS+BH(puncture): non-public





Initial Equilibrium Stars (for NS & SMS)

(Komatsu, Eriguchi, & Hachisu, KEH89)

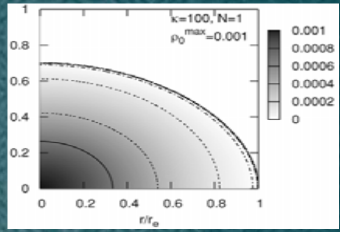
$$ds^2 = -e^{2\nu}dt^2 + e^{2\alpha}\left(dr^2 + r^2 d\theta^2\right) + e^{2\beta} r^2 \sin^2\theta \left(d\phi - \omega dt\right)^2 .$$

- Perfect fluid e.m. tensor, T^{ab}
- Fluid four-velocity, u^a
- Proper velocity w.r.t zamo, v
- angular velocity measured from infinity, Ω

$$T^{ab} = (\varepsilon + p) u^a u^b + p g^{ab}$$

$$u^a = \frac{e^{-\nu}}{\sqrt{1-v^2}}(1,0,0,\Omega)$$

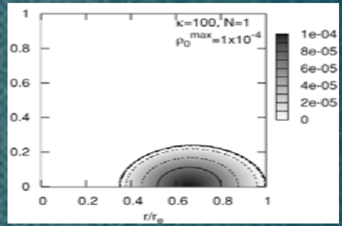
$$v = (\Omega - \omega) r \sin\theta e^{\beta-\nu}$$



$$\nabla p + (\varepsilon + p)\left[\nabla\nu + \frac{1}{1-v^2}\left(-v\nabla v + v^2 \frac{\nabla\Omega}{\Omega-\omega}\right)\right] = 0$$

- Hydrostatic eq is integrable if $\frac{v^2}{(1-v^2)(\Omega-\omega)} = j(\Omega)$

$$(1+N)\ln(K\varepsilon^{1/N}+1) + \nu + \tfrac{1}{2}\ln(1-v^2) + \int j(\Omega)\, d\Omega = 0$$



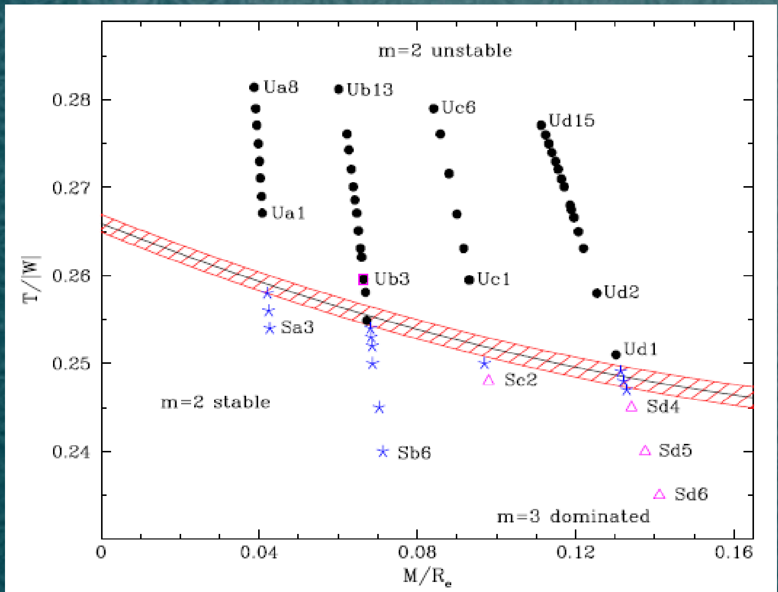
- $j(\Omega)$ is given by hands $\quad j(\Omega) = A^2(\Omega_c - \Omega)$
- H is entalphy $\quad \ln H \equiv C - \nu - \tfrac{1}{2}\ln(1-v^2) + \tfrac{1}{2}A^2(\Omega - \Omega_c)^2$

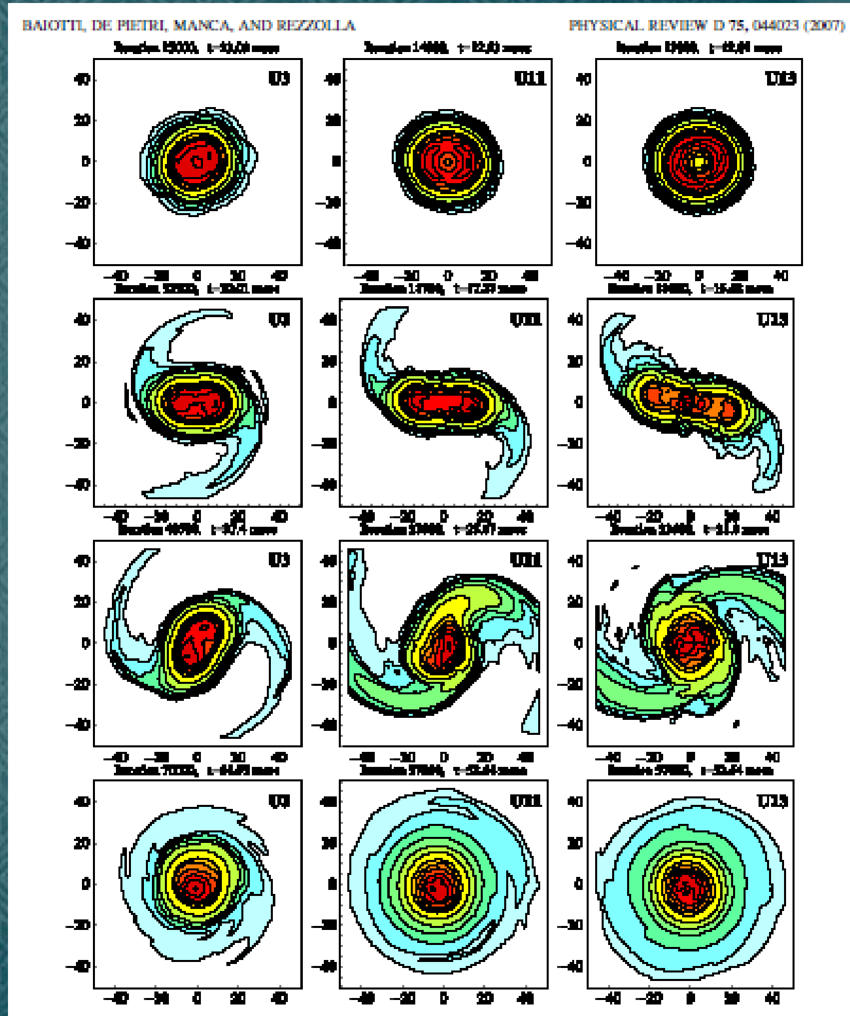
- Iteration: ρ → metrics → H with b.c. (H=0) → ρ
- Parameters: axis ratio, maximum density, rotation parameter A

Topics of rotating stars
(Non-axisymmetric instability)

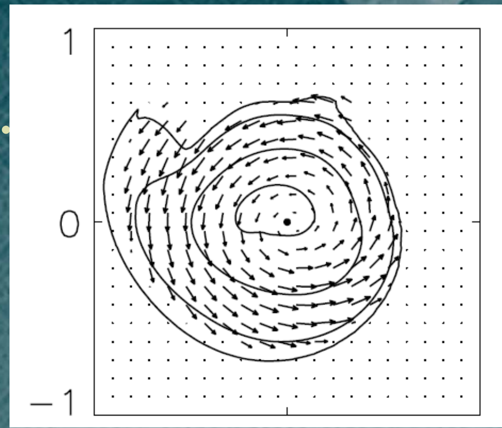
- Bar-mode instability
  - Baiotti etal, Manca etal 07
  - Low central density

$$\beta_c = 0.266 - 0.18\left(\frac{M}{R_e}\right) + 0.36\left(\frac{M}{R_e}\right)^2$$



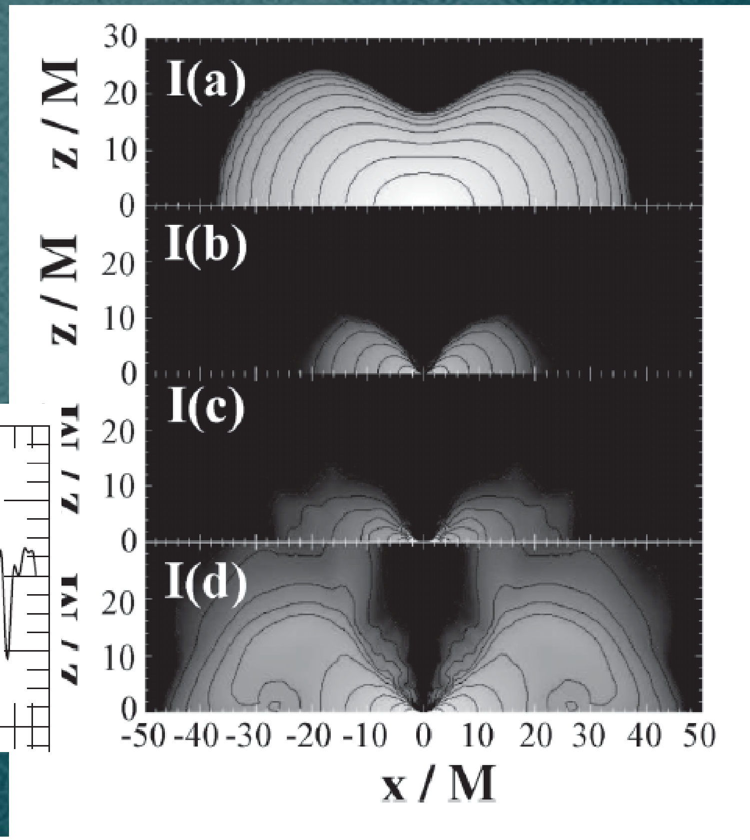


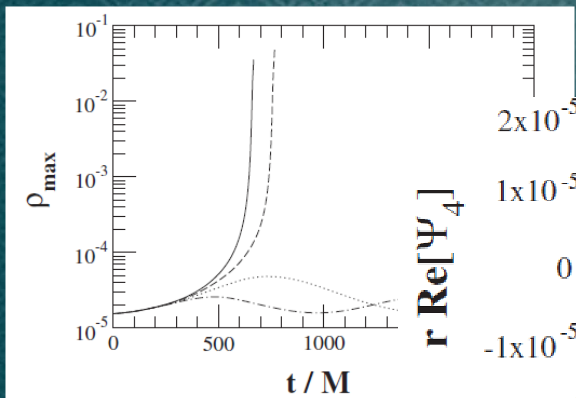
- Low-β instability (Ou & Tohline 06)
  - Bar formation even for β ~ 0.01
  - strongly differential rotation
  - shear instability
  - Corvino etal 2010 with SLy

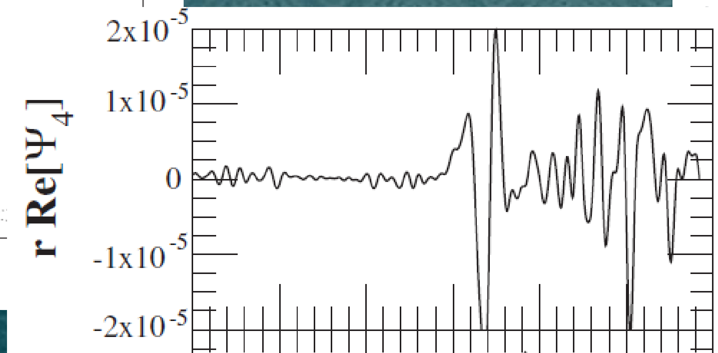


- Collapse of Supermassive Star (Saijo & Hawke 09)

  - Bar formation even for β ~ 0.01
  - strongly differential rotation
  - shear instability
  - Corvino etal 2010 with SLy





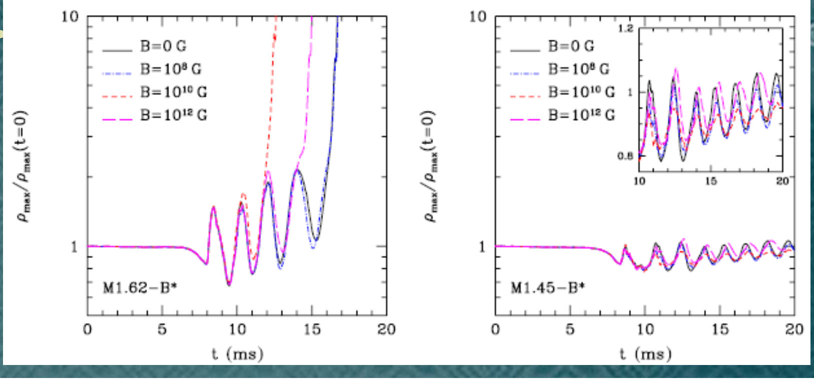


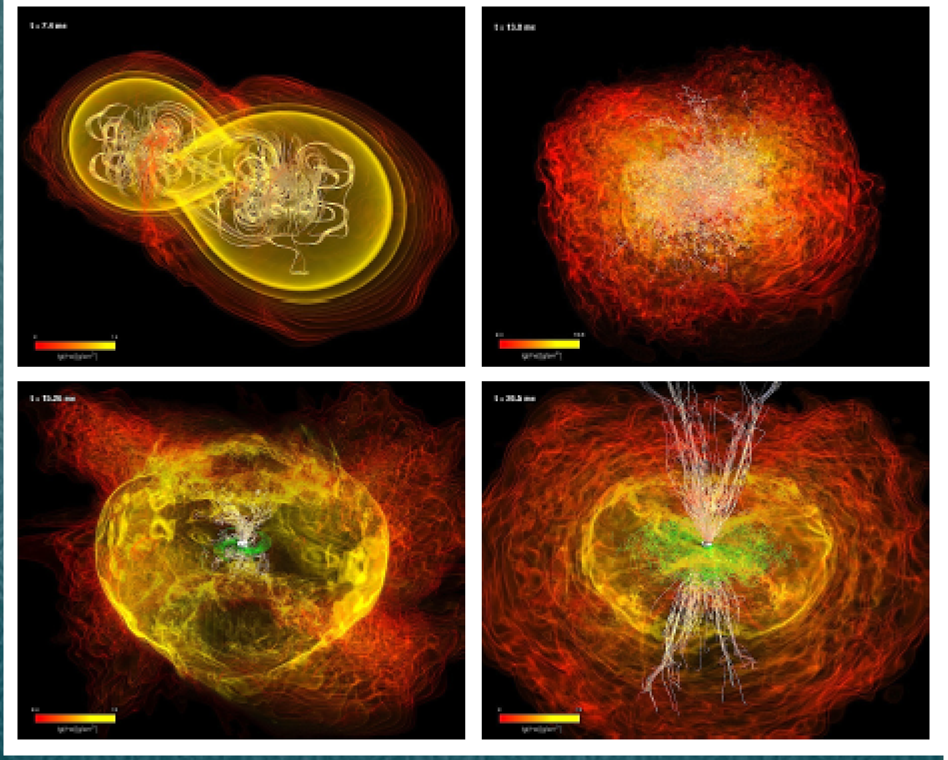
Binary NS Merge

- High mass binary
  - Prompt BH formation
  - BH + Torus
- Low mass binary
  - Delayed BH formation
  - Intermediate Hyper Massive Neutron Star
- Temperature rises up to 10^11-12 K
  - Cooling should be considered
- Magnetic field growth

Binary NS (MHD)

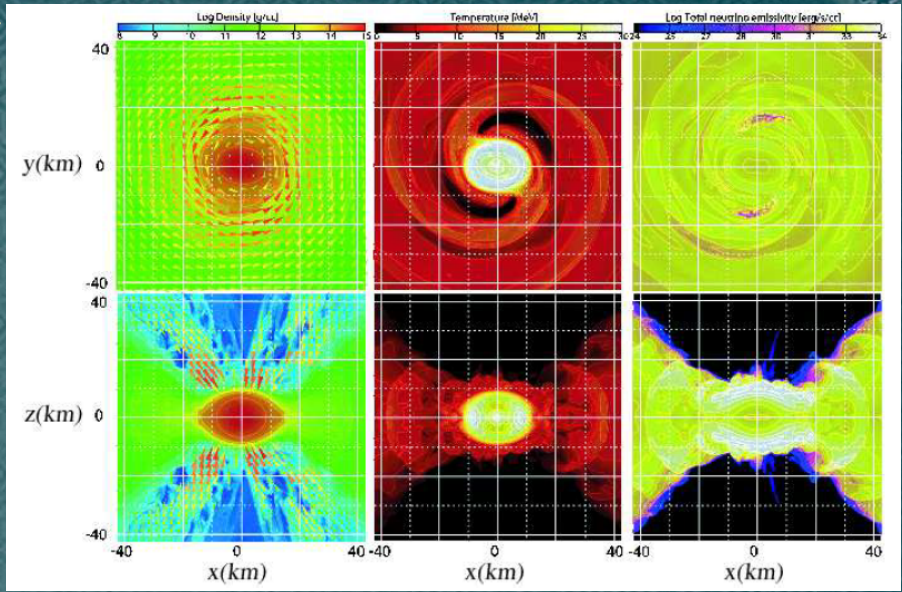
- Whisky group
  - Giacomazzo, Rezzolla, Baiotti 2011
  - Simple Ideal EoS
  - But highest res.
- Short GRB
  - Rezzolla etal 2011
  - M_g = 1.5
  - Initial 10^12 G → 10^15G
  - Opening angle 30 degree
  - Cooling is not considered

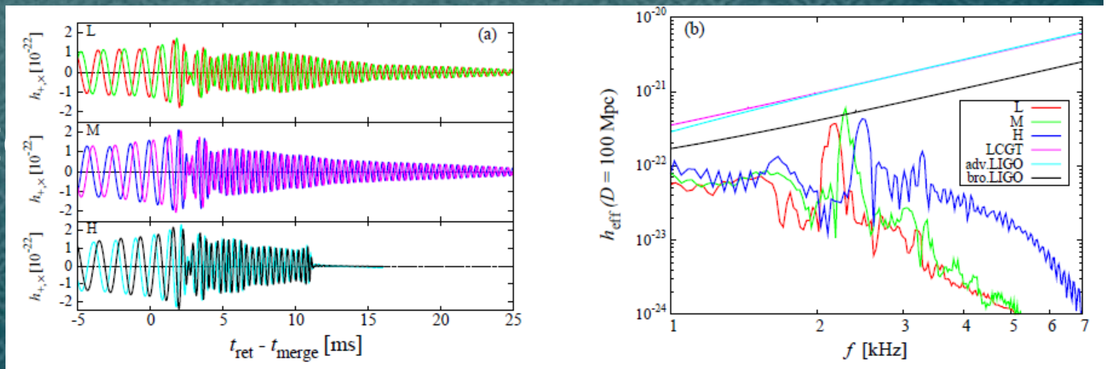




Binary NS: Neutrino transport

- Sekiguchi etal 2011
- 1st NR result for binary merge with neutrino cooling calculation
- Shen's hot EoS
- If M<3.2M_solar
  - No prompt BH formation
- Neutrino emission rat ~ 3-8x10^53 ergs/s
- GWs at 2.1-2.5 kHz





Concluding Remarks

- What is possible in current GRHydro
  - accurate & stable HRSC scheme
  - good initial data for single/binary NS
  - realistic EOSs
  - ideal MHD (cf. resistive MHD of AEI)
  - Neutrino transport even in binary NS merges
- Future tasks of GRHydro
  - better initial data, initial data for pulsars
  - better numerical schemes
  - full understandings on MHD
  - better neutrino transport / EM processes